Precise Positioning of a Sea Bottom Transponder (4)

—Effects of Current on Positioning Precision (1)—

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Summary

In the previous papers, it is shown that not only the position of a bottom transponder but also the underwater acoustic velocity distribution can be determined, if the position of surface transponders and ranges between the surface and bottom transponders are known. Also, a theoretical possibility was shown that the position of the bottom transponder can be obtained in the order of centimeters by using equipments available now. They were extended to three-dimensions in the third paper. More realistic results were obtained there. In the present paper, the theory is extended to include the effects of current. The basic nonlinear equations are obtained by a variational principle. Some two-dimensional numerical calculations are conducted, and the convergence of the iteration calculations is verified. The numerical results show that the effects of current can be obtained and can't be neglected for precise positioning.

1. Introduction

In the first report, if the position of sea surface transponders and the distance between the surface and bottom transponders are given, not only the position of a sea bottom transponder but also the underwater acoustic velocity can be determined. A division into two or three horizontal layers seems to be sufficient for the estimation of the correct position of the bottom transponder.

In the second report, a theoretical possibility of obtaining the position of the sea bottom transponder with accuracy of centimeter order by using measuring instruments available at present is shown. For the precise estimation of the bottom transponder position, the simultaneous estimation of the underwater acoustic velocity distribution and the position of the underwater transponder is shown to be extremely important.

In the first and second papers, two-dimensional calculations are conducted. In the third paper, more realistic simulations are conducted by three-dimensional calculations. And a new idea called MIL (Method of Incremental Layers) is introduced to stabilize numerical calculations. Specifically, the number of layers is increased one by one. The initial value of the iteration calculation is obtained from the previous layer division. It is confirmed that MIL improves the stability of convergence calculation.

In the previous reports, the current velocity was taken zero. In the present paper, not only the effect of the underwater acoustic velocity distribution but also those of the current velocity distribution is discussed. The basic equations are obtained, and some numerical calculations are also conducted.

2. Basic Equations

Let x and y be horizontal axes and z vertical axis as shown in Fig. 1. The underwater acoustic velocity and current are supposed to vary as $C(z)$ and $(U_x(z), U_y(z), 0)$ for $-h \le z \le 0$, where h is water depth. The acoustic ray $x = x(z),\ y = y(z),\ (z_a \ge z \ge z_b)$ connecting points $P_a(x_a, y_a, z_a)$ and $P_b(x_b, y_b, z_b)$ is solution of a variational problem [4]:

$$t[x,y] = \int_{z_a}^{z_b} \frac{ds}{C(z) - [U_x(dx/ds) + U_y(dy/ds)]}$$
$$= -\int_{z_a}^{z_b} \left[(dx/dz)^2 + (dy/dz)^2 + 1\right] dz \Big/ \Big[C(z) \sqrt{(dx/dz)^2 + (dy/dz)^2 + 1} - U_x(dx/dz) - U_y(dy/dz)\Big]$$
$$= \min \qquad (1a)$$

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$$x(z_a) = x_a = given, \quad y(z_a) = y_a = given,$$
$$x(z_b) = x_b = given, \quad y(z_b) = y_b = given \tag{1b}$$

where the functional $t[x, y]$ is time necessary for the acoustic wave to travel from P_a to P_b, and ds is $\sqrt{dx^2 + dy^2 + dz^2}$.

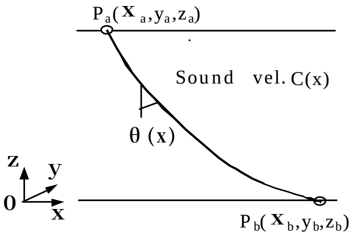

Fig. 1 Sound propagating underwater

The Euler equations of the variational problem (1) give

$$\frac{\left[-C(z)\sin q_x(z) + U_x(z)\cos 2q_x(z) - 2U_y(z)\sin q_x(z)\sin q_y(z)\right]}{\left[C(z)+U_x(z)\sin q_x(z)+U_y(z)\sin q_y(z)\right]^2} = const \tag{2a}$$

$$\frac{\left[-C(z)\sin q_y(z) + U_y(z)\cos 2q_y(z) - 2U_x(z)\sin q_x(z)\sin q_y(z)\right]}{\left[C(z)+U_x(z)\sin q_x(z)+U_y(z)\sin q_y(z)\right]^2} = const \tag{2b}$$

where

$$\sin q_x(z) = dx/ds \tag{3a}$$

$$\sin q_y(z) = dy/ds \tag{3b}$$

Eq. (3) is nothing but Snell's law. If angle between vectors $d\mathbf{s}$ and $d\mathbf{z}$ is denoted by l and angle between vectors $d\mathbf{s}-d\mathbf{z}$ and $d\mathbf{x}$ by m, relation between q_x, q_y and l, m is given as

$$dx/ds = \sin q_x(z) = \sin l \cos m \tag{4a}$$

$$dy/ds = \sin q_y(z) = \sin l \sin m \tag{4b}$$

Various problems are born depending on whether the acoustic velocity C, current velocity U, inclination angle q of acoustic rays and bottom transponder position are considered known or unknown. The problems are classified in Table 1.

3. A Numerical Procedure for Two-Dimensional Problems

In the following, slowness $S(z)$ is used instead of sound velocity $C(z)$:

$$S(z) = 1/C(z) \tag{5}$$

As shown in Fig. 2, the acoustic field is divided into horizontal layers to discretize the problem. The discretized equations are shown below. In the following equations, m and n refer to ray and layer.

Snell's Law:

$$\frac{S_n\left(\sin q_{mn} - S_n U_n \cos 2q_{mn}\right)}{\left(1+S_n U_n \sin q_{mn}\right)^2} = \frac{S_{n+1}\left(\sin q_{m\,n+1} - S_{n+1}U_{n+1}\cos 2q_{m\,n+1}\right)}{\left(1+S_{n+1}U_{n+1}\sin q_{m\,n+1}\right)^2}$$
$$(m = 0,1,\cdots,M-1;\ n = 1,2,\cdots,N-1) \tag{6}$$

Travel time of sound signal:

$$\Delta t_m = \sum_{n=0}^{N-1}\Delta t_{mn} = \sum_{n=0}^{N-1}\frac{h_n}{\cos q_{mn}}\frac{1}{C_n - U_n \sin q_{mn}} = \sum_{n=0}^{N-1}\frac{h_n}{\cos q_{mn}}\frac{S_n}{1 - S_n U_n \sin q_{mn}},$$
$$(m = 0,1,\cdots,M-1) \tag{7}$$

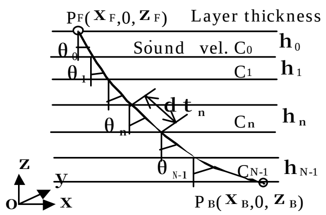

Fig. 2 Approximation of sound field

Table 1 Classification of problems

No.	Case Name	Acoustic Velocity C	Current Velocity U	Inclination Angle q	Position of Transponder x_B
1	SUTP1101	given	given	unknown	Given
2	SUTP1100	given	given	unknown	unknown
3	SUTP1000	given	unknown	unknown	unknown
4	SUTP0100	unknown	given	unknown	unknown
5	SUTP0000	unknown	unknown	unknown	unknown

Horizontal distance between the surface and bottom transponders:

$$x_B - x_{Fm} = \sum_{n=0}^{N-1} h_n \tan\theta_{mn}, \quad (m = 0,1,\cdots, M-1) \qquad (8)$$

The Eqs. (6), (7) and (8) form a closed system. Namely, the number of unknowns θ_{mn}, C_n, U_n, x_B; $m = 0,\cdots, M-1$, $n = 0,\cdots, N-1$ is $MN + 2N + 1$. On the other hand, the number of equations is $M(N-1) + 2M$. Hence, if

$$M = 2N + 1, \qquad (9)$$

the number of the unknowns coincides with that of the equation.

Snell's Law given by Eq. (2) can be written as

$$\frac{C(z)\sin\theta_x(z) - U_x(z)\cos 2\theta_x(z)}{\left[C(z) + U_x(z)\sin\theta_x(z)\right]^2} = k_x(x_F) \qquad (10)$$

if the x-coordinate of the acoustic ray and the constant of the ray are denoted as x_F and $k_x(x_F)$ respectively. This equation gives relationship between $\theta_x(z)$, $C(z)$, $U_x(z)$ and $k_x(x_F)$. Hence, the unknowns $\theta_x(z)$ can be replaced by the new unknowns $k_x(x_F)$. An algorithm to eliminate $\theta_x(z)$ is given in Appendix.

The nonlinear equations (6) through (8) are solved by Newton-Raphson Method as follows.

Snell's Law:

$$\frac{S_n\left(\sin\theta_{mn} - S_nU_n\cos 2\theta_{mn}\right)}{\left(1+S_nU_n\sin\theta_{mn}\right)^2}$$

$$+\left[\frac{\left(\sin\theta_{mn} - 2S_nU_n\cos 2\theta_{mn}\right)}{\left(1+S_nU_n\sin\theta_{mn}\right)^2} - 2S_n\left(\sin\theta_{mn} - S_nU_n\cos 2\theta_{mn}\right)\cdot\frac{\left(1+S_nU_n\sin\theta_{mn}\right)U_n\sin\theta_{mn}}{\left(1+S_nU_n\sin\theta_{mn}\right)^4}\right]dS_n$$

$$+\left[\frac{S_n\left(\cos\theta_{mn} + 2S_nU_n\sin 2\theta_{mn}\right)}{\left(1+S_nU_n\sin\theta_{mn}\right)^2} - 2S_n\left(\sin\theta_{mn} - S_nU_n\cos 2\theta_{mn}\right)\cdot\frac{\left(1+S_nU_n\sin\theta_{mn}\right)S_nU_n\cos\theta_{mn}}{\left(1+S_nU_n\sin\theta_{mn}\right)^4}\right]d\theta_{mn}$$

$$+\left[\frac{-S_n^{\,2}\cos 2\theta_{mn}}{\left(1+S_nU_n\sin\theta_{mn}\right)^2} - 2S_n\left(\sin\theta_{mn} - S_nU_n\cos 2\theta_{mn}\right)\cdot\frac{\left(1+S_nU_n\sin\theta_{mn}\right)S_n\sin\theta_{mn}}{\left(1+S_nU_n\sin\theta_{mn}\right)^4}\right]dU_n$$

$$=\frac{S_{n+1}\left(\sin\theta_{m\,n+1} - S_{n+1}U_{n+1}\cos 2\theta_{m\,n+1}\right)}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^2}$$

$$+\left[\frac{\left(\sin\theta_{m\,n+1} - 2S_{n+1}U_{n+1}\cos 2\theta_{m\,n+1}\right)}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^2} - 2S_{n+1}\left(\sin\theta_{m\,n+1} - S_{n+1}U_{n+1}\cos 2\theta_{m\,n+1}\right)\cdot\frac{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)U_{n+1}\sin\theta_{m\,n+1}}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^4}\right]dS_{n+1}$$

$$+\left[\frac{S_{n+1}\left(\cos\theta_{m\,n+1} + 2S_{n+1}U_{n+1}\sin 2\theta_{m\,n+1}\right)}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^2} - 2S_{n+1}\left(\sin\theta_{m\,n+1} - S_{n+1}U_{n+1}\cos 2\theta_{m\,n+1}\right)\cdot\frac{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)S_{n+1}U_{n+1}\cos\theta_{m\,n+1}}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^4}\right]d\theta_{m\,n+1}$$

$$+\left[\frac{-S_{n+1}^{\,2}\cos 2\theta_{m\,n+1}}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^2} - 2S_{n+1}\left(\sin\theta_{m\,n+1} - S_{n+1}U_{n+1}\cos 2\theta_{m\,n+1}\right)\cdot\frac{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)S_{n+1}\sin\theta_{m\,n+1}}{\left(1+S_{n+1}U_{n+1}\sin\theta_{m\,n+1}\right)^4}\right]dU_n,$$

$$(m = 0,1,\cdots, M-1;\; n = 1,2,\cdots, N-1) \qquad (11)$$

Travel time of acoustic signal:

$$\Delta t_m = \sum_{n=1}^{N}\frac{h_n}{\cos\theta_{mn}}\frac{S_n}{1-S_nU_n\sin\theta_{mn}}$$

$$+\sum_{n=1}^{N}\left\{\left[\frac{h_n}{\cos\theta_{mn}}\frac{1}{1-S_nU_n\sin\theta_{mn}} + \frac{h_n}{\cos\theta_{mn}}\frac{S_nU_n\sin\theta_{mn}}{\left(1-S_nU_n\sin\theta_{mn}\right)^2}\right]dS_n\right.$$

$$+\left[\frac{h_n\sin\theta_{mn}}{\cos^2\theta_{mn}}\frac{S_n}{1-S_nU_n\sin\theta_{mn}} + \frac{h_n}{\cos\theta_{mn}}\frac{S_n^{\,2}U_n\cos\theta_{mn}}{\left(1-S_nU_n\sin\theta_{mn}\right)^2}\right]d\theta_{mn}$$

$$\left.+\left[\frac{h_n}{\cos\theta_{mn}}\frac{S_n^{\,2}\sin\theta_{mn}}{\left(1-S_nU_n\sin\theta_{mn}\right)^2}\right]dU_n\right\},$$

$$(m = 0,1,\cdots, M-1) \qquad (12)$$

Horizontal distance between the surface and bottom transponders:

$$x_B - x_{Fm} + dx_B = \sum_{n=1}^{N} h_n\tan\theta_{mn} + \sum_{n=1}^{N}\frac{h_n}{\cos^2\theta_{mn}}d\theta_{mn},$$

$$(m = 0,1,\cdots, M-1) \qquad (13)$$

4. Numerical Results in Two-Dimensional Problems

Distribution of underwater sound and current velocity used in numerical calculations are shown in Fig. 3. The water depth is 1000m, and the region is divided into 40 layers.

In the following, Eqs. (11) through (13) are used. First, Problem 2 in Table 1 was solved. In this problem, the sound and current velocities are assumed given. The number of unknowns

θ_{mn}, x_B; $m = 1,\cdots,M$, $n = 1,\cdots,N$ is $MN+1$, and that of equations is $M(N-1)+2M$. Hence, if

$$M = 1, \qquad (14)$$

the number of the unknowns coincides that of the equations. When the number of the acoustic rays are bigger than one, the least square procedure is applied.

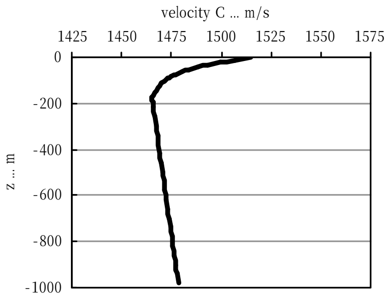

Fig. 3a Distribution of acoustic velocity C

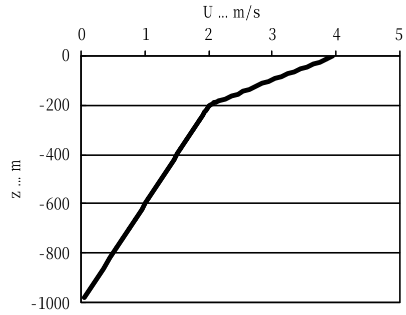

Fig. 3b Distribution of current velocity U

In the following, five rays are used. The characteristics of the five rays are given in Table 2. The x-coordinate of the bottom transponder x_B is assumed zero.

Table 2 Characteristics of sound rays

	Ray 0	Ray 1	Ray 2	Ray 3	Ray 4
Pos. of Surface Transponder	-448.880m	-669.563	-943.529	-1311.899	-1854.256
One Way Travel Time	0.744288s	0.817281	0.933793	1.120475	1.431079
Incidence Angle of Ray*	25deg	35	45	55	65
Average	24.174	33.804	43.334	52.679	61.650

* The angle between the ray and the negative z direction at the surface.

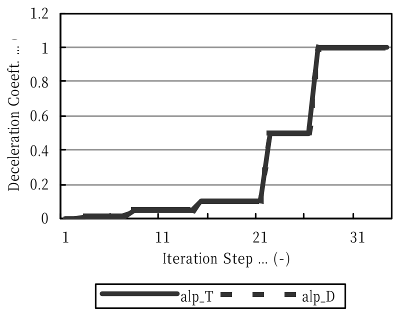

Fig 4a Selection of deceleration coefficients α (SUTP1100; 40 layers)

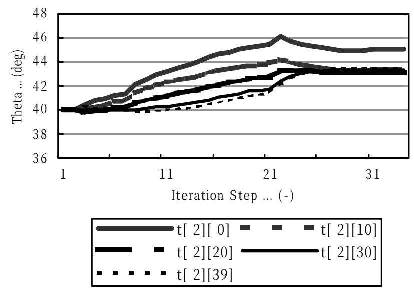

Fig. 4b Convergence of ray inclination angles θ (SUTP1100; 40 layers; t[m][n] denotes θ_{mn})

Fig. 4 shows the results for a case where the underwater field is divided into 40 layers. α s are deceleration coefficients in iteration procedure[2)-4)]. As shown in Fig. 4a, α for θ_{mn} (i.e. alp_T) and that for x_B (i.e. alp_D) should be different in general. The convergence of the iteration procedure was very nice in spite of the big number of the layer division. As can be seen from Fig. 4e, the effect of the current velocity on the ray trajectory seems not significant, but the effect can't be neglected for the precise positioning as premised in the present paper as shown in Fig. 4f.

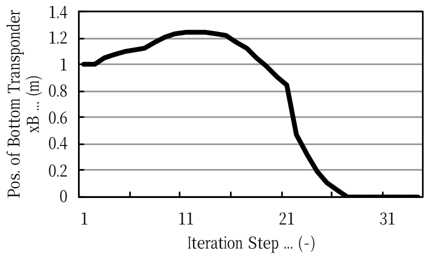

Fig. 4c Convergence of horizontal position of bottom transponder x_B (SUTP1100; 40 layers)

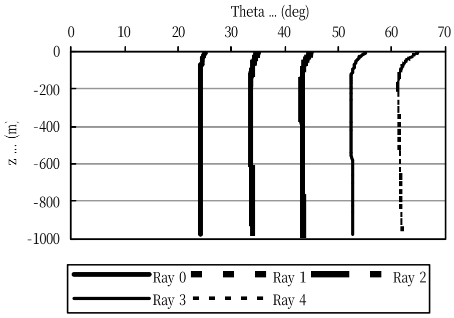

Fig. 4d Convergence of ray inclination angles θ
(SUTP1100; 40 layers)

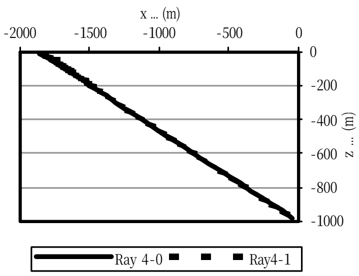

Fig. 4e Effect of current velocity on ray trajectory
(SUTP1100; 40 layers; Ray 4-0 and Ray-1
denote with and without current)

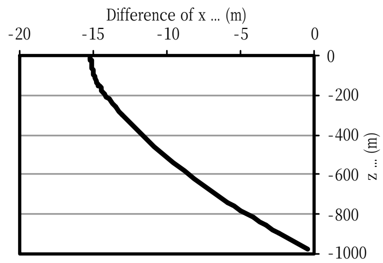

Fig. 4f Difference between ray trajectories
with and without current
(SUTP1100; 40 layers; Ray 4)

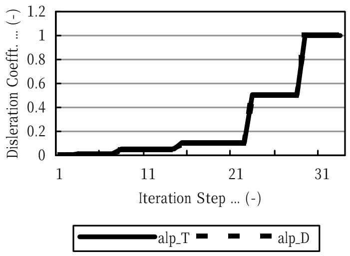

Fig 5a Selection of deceleration coefficients α
(SUTP1100; 1 layer)

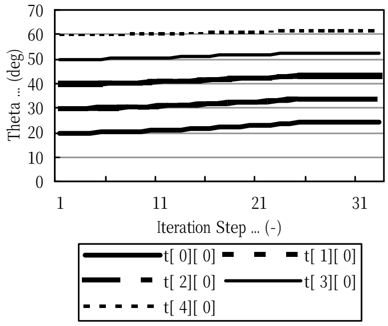

Fig. 5b Convergence of ray inclination angles θ
(SUTP1100; 1 layer; t[m][n] denotes θ_{mn})

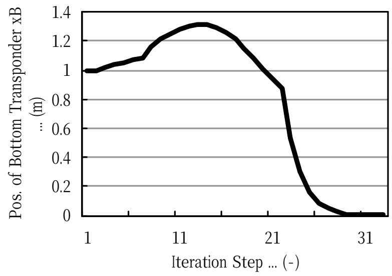

Fig. 5c Convergence of horizontal position
of bottom transponder x_B
(SUTP1100; 1 layer)

Fig. 5 shows the results for a case where the underwater field is divided into 1 layer. A comparison of the acoustic rays and the horizontal position of the bottom transponder between 40 layers and 1 layer divisions is made in Table 3.

The one layer solution gives a very nice approximation unexpectedly. The average slopes of rays in case of 40 layers division agree very well with those in case of 1 layer division. The horizontal position of the bottom transponder x_B in case of 1 layer division is 0.007m, where the correct value is 0m.

Table 3 A comparison of the acoustic rays and the horizontal position of the bottom transponder between 40 layers and 1 layer divisions (SUTP1100)

		Ray 0	Ray 1	Ray 2	Ray 3	Ray 4	x_B
40 layers	Incidence	25deg	35	45	55	65	0m
	Average	24.174	33.804	43.334	52.679	61.650	
1 layer		24.175	33.805	43.336	52.684	61.662	0.007

Table 4 The sound velocity, the current velocity, the rays and the horizontal position of the bottom transponder (SUTP0000; 1 layer)

	C	U	Ray 0	Ray 1	Ray 2	Ray 3	Ray 4	x_B
40 layer Average	1473.33 m/s	1.4 m/s	24.174 deg	33.804	43.334	52.679	61.650	0m
1 layer	1473.3	1.3	24.183	33.812	43.341	52.687	61.664	0.17

Since the sound and current velocities may be unknown in many cases, the results in these situations are shown in Fig. 6. In Fig. 6a, alp_S, alp_T, alp_U and alp_D denote the deceleration coefficients for S_m, U_m θ_{mn} and x_B .The sound velocity, the current velocity, the rays and the horizontal position of the bottom transponder are summarized in Table 4. In this calculation, the layer division is one. The results seem to be reasonable, although the precision of x_B is not sufficient. To increase the accuracy, the number of the layers must be increased. This problem will be discussed in the next report.

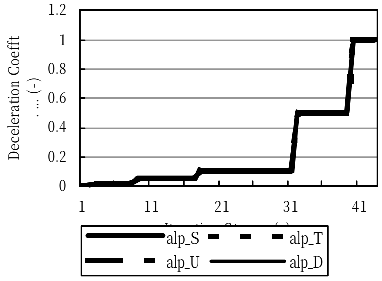

Fig 6a Selection of deceleration coefficients α (SUTP0000; 1 layer)

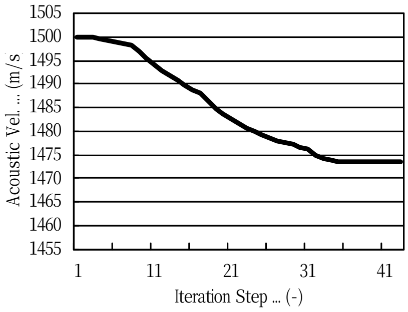

Fig. 6b Convergence of sound velocity C (SUTP000; 1 layer)

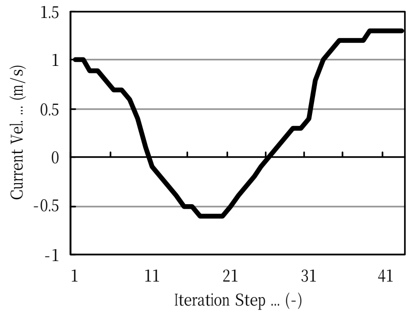

Fig. 6c Convergence of current velocity U (SUTP0000; 1 layer)

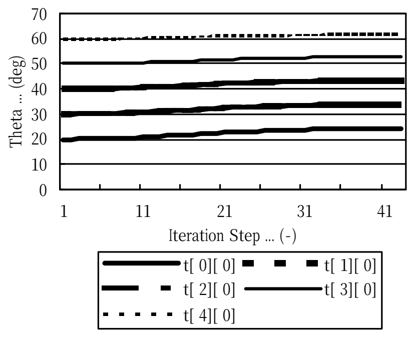

Fig. 6d Convergence of ray inclination angles θ (SUTP0000; 1 layer; t[m][n] denotes θ_{mn})

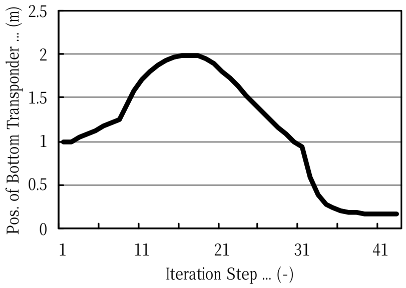

Fig. 6e Convergence of horizontal position of bottom transponder x_B (SUTP0000; 1 layer)

5. Conclusion

In the previous reports, the current velocity was assumed zero. In the present report, the theory to include the effects of the current was constructed, and some two-dimensional numerical calculations were conducted. According to the results, the effects of the current can't be neglected to realize a high precision positioning of centimeter order necessary to the measurements of the sea bottom crust movements.

To introduce the effects of the currents fully, many studies must be done. When the acoustic and current velocities are unknowns, the most important problem is how to solve the difficulty in convergence of iterations as the number of the layers increases. The results will be reported in the coming reports.

On the other hand, an application of the present theory to a field other than the measurements of the sea bottom crust movements may also be pursued.

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References

1) H. Isshiki, H. Endo and Y. Terada, "Precise positioning of a transponder placed on a seafloor", 2000 Underwater Technology held at Tokyo, (2000/May).

2) H. Isshiki, "Precise Positioning of a Sea Bottom Transponder (1)", Journ. of the Society of Naval Architects of Japan, Vol. 188, (2000).

3) H. Isshiki, "Precise Positioning of a Sea Bottom Transponder (2) －Principle of Measurements and Error Analysis－", Journ. of the Society of Naval Architects of Japan, Vol. 189, (2001).

4) H. Isshiki, "Precise Positioning of a Sea Bottom Transponder (3) －Three Dimensional Calculation－", Journ. of the Society of Naval Architects of Japan, Vol. 190, (2001).

5) D. Chadwell, F. Spiess, J. Hildebrand, L. Young, G. Purcell, Jr. and H. Dragert, "Deep-sea geodesy: Monitoring the ocean floor", GPS World, (1998/Sept).

6) F. N. Spiess, C. D. Chadwell, J. A. Hildebrand, L. E. Young, G. H. Purcell Jr. and H. Dragert, "Precise GPS/Acoustic positioning of seafloor reference points for tectonic studies", Physics of the Earth and Planetary Interiors, 108, (1998), pp. 101-112.

Appendix An algorithm to eliminate $q_x(z)$

Let $C(z)$, $U(z)$ and $k_x(x_F)$ be given. Then, $q_x(z)$ is obtained by solving the two-dimensional form of Eq. (2a):

$$\frac{C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^2} = k_x(x_F) \qquad \text{(A1)}$$

Since Eq. (A1) can't be solved analytically, a numerical solution is obtained by using Newton-Raphson method:

$$\frac{C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^2} + \left\{ \frac{[C(z)\cos q_x(z) + 2U_x(z)\sin 2q_x(z)]}{[C(z)+U_x(z)\sin q_x(z)]^2} - 2[C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)]\cdot [C(z)+U_x(z)\sin q_x(z)]\cdot \frac{U_x(z)\cos q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^4} \right\} dq_x(z) = k_x(x_F) \qquad \text{(A2)}$$

The dependence between $q_x(z)$, $C(z)$, $U(z)$ and $k_x(x_F)$ are obtaines by differentiating Eq. (A1) as

$$\left\{ \frac{[C(z)\cos q_x(z) + 2U_x(z)\sin 2q_x(z)]}{[C(z)+U_x(z)\sin q_x(z)]^2} - 2[C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)]\cdot \frac{U_x(z)\cos q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^3} \right\} dq_x(z)$$
$$+ \left\{ \frac{\sin q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^2} - \frac{2[C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)]}{[C(z)+U_x(z)\sin q_x(z)]^3} \right\} dC(z)$$
$$+ \left\{ -\frac{\cos 2q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^2} - 2[C(z)\sin q_x(z) - U_x(z)\cos 2q_x(z)]\cdot \frac{\sin q_x(z)}{[C(z)+U_x(z)\sin q_x(z)]^3} \right\} dU_x(z)$$
$$= dk_x(x_F) \qquad \text{(A3)}$$

Eqs. (A1) and (A3) can be used to eliminate $q_x(z)$.