Precise Positioning of a Sea Bottom Transponder (4)  
- Effects of Current on Positioning Precision (1) -

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Summary

In the previous papers, it is shown that not only the position of a bottom transponder but also the underwater acoustic velocity distribution can be determined, if the position of surface transponders and ranges between the surface and bottom transponders are known. Also, a theoretical possibility was shown that the position of the bottom transponder can be obtained in the order of centimeters by using equipments available now. They were extended to three-dimensions in the third paper. More realistic results were obtained there. In the present paper, the theory is extended to include the effects of current. The basic nonlinear equations are obtained by a variational principle. Some two-dimensional numerical calculations are conducted, and the convergence of the iteration calculations is verified. The numerical results show that the effects of current can be obtained and can’t be neglected for precise positioning.

1. Introduction

In the first report, if the position of sea surface transponders and the distance between the surface and bottom transponders are given, not only the position of a sea bottom transponder but also the underwater acoustic velocity can be determined. A division into two or three horizontal layers seems to be sufficient for the estimation of the correct position of the bottom transponder.

In the second report, a theoretical possibility of obtaining the position of the sea bottom transponder with accuracy of centimeter order by using measuring instruments available at present is shown. For the precise estimation of the bottom transponder position, the simultaneous estimation of the underwater acoustic velocity distribution and the position of the underwater transponder is shown to be extremely important.

In the first and second papers, two-dimensional calculations are conducted. In the third paper, more realistic simulations are conducted by three-dimensional calculations. And a new idea called MIL (Method of Incremental Layers) is introduced to stabilize numerical calculations. Specifically, the number of layers is increased one by one. The initial value of the iteration calculation is obtained from the previous layer division. It is confirmed that MIL improves the stability of convergence calculation.

In the previous reports, the current velocity was taken zero. In the present paper, not only the effect of the underwater acoustic velocity distribution but also those of the current velocity distribution is discussed. The basic equations are obtained, and some numerical calculations are also conducted.

2. Basic Equations

Let $x$ and $y$ be horizontal axes and $z$ vertical axis as shown in Fig. 1. The underwater acoustic velocity and current are supposed to vary as $C(z)$ and $(U_x(z), U_y(z), 0)$ for $-h \leq z \leq 0$, where $h$ is water depth. The acoustic ray $x = x(z), y = y(z), (z_0 \geq z \geq z_b)$ connecting points $P_a(x_a, y_a, z_a)$ and $P_b(x_b, y_b, z_b)$ is solution of a variational problem:

$$\zeta[x, y] = \int_a^b \frac{ds}{C(z) - \sqrt{U_x(dz/ds)^2 + U_y(dz/ds)^2}}$$

$$= -\int_a^b \left[ \frac{dz}{\sqrt{dz/df(z)^3 + (dz/df(z))^2 + 1}} \right] dz$$

$$= \min_{\text{under}}$$

\begin{equation}
\tag{1a}
\end{equation}
where the functional \( I(x, y) \) is time necessary for the acoustic wave to travel from \( P_a \) to \( P_b \), and \( ds \) is
\[
\sqrt{dx^2 + dy^2 + dz^2}.
\]

Fig. 1 Sound propagating underwater

The Euler equations of the variational problem (1) give
\[
\begin{align*}
-C(z) \sin \theta_1(z) + U_1(z) \cos 2\theta_1(z) \\
-2U_1(z) \sin \theta_1(z) \sin \theta_1(z) \\
\left[ C(z) + U_1(z) \sin \theta_1(z) + U_1(z) \sin \theta_1(z) \right]^{1/2}
\end{align*}
\]
\[
= \text{const}
\]
\[ (2a) \]
\[
\begin{align*}
-C(z) \sin \theta_1(z) + U_1(z) \cos 2\theta_1(z) \\
-2U_1(z) \sin \theta_1(z) \sin \theta_1(z) \\
\left[ C(z) + U_1(z) \sin \theta_1(z) + U_1(z) \sin \theta_1(z) \right]^{1/2}
\end{align*}
\]
\[
= \text{const}
\]
\[ (2b) \]
where
\[
\sin \theta_1(z) = \frac{dx}{ds}
\]
\[
\sin \theta_1(z) = \frac{dy}{ds}
\]
Eq. (3) is nothing but Snell’s law. If angle between vectors \( ds \) and \( dz \) is denoted by \( \lambda \) and angle between vectors \( ds - dx \) and \( dx \) by \( \mu \), relation between \( \theta_1, \theta_2, \) and \( \lambda, \mu \) is given as
\[
dx = \sin \theta_1(z) = \sin \lambda \cos \mu
\]
\[
dy = \sin \theta_2(z) = \sin \lambda \sin \mu
\]

Various problems are born depending on whether the acoustic velocity \( C \), current velocity \( U \), inclination angle \( \theta \) of acoustic rays and bottom transponder position are considered known or unknown. The problems are classified in Table 1.

### 3. A Numerical Procedure for Two-Dimensional Problems

In the following, slowness \( S(z) \) is used instead of sound velocity \( C(z) \):
\[
S(z) = \frac{1}{C(z)}
\]
As shown in Fig. 2, the acoustic field is divided into horizontal layers to discretize the problem. The discretized equations are shown below. In the following equations, \( m \) and \( n \) refer to ray and layer.

Snell’s Law:
\[
S_n \left( \sin \theta_{mn} - S_n U_n \cos 2\theta_{mn} \right) \left( 1 + S_n U_n \sin \theta_{mn} \right)
\]
\[
= \sum_{m=0}^{N-1} S_m \left( \sin \theta_{mn} - S_n U_n \cos 2\theta_{mn} \right) \left( 1 + S_n U_n \sin \theta_{mn} \right)
\]
\[ (m = 0, 1, \ldots, M - 1; \ n = 1, 2, \ldots, N - 1) \]
\[ (6) \]

Travel time of sound signal:
\[
\Delta t_m = \sum_{n=0}^{N-1} \Delta t_{mn} = \sum_{n=0}^{N-1} \frac{h_n}{\cos \theta_{mn} C_n - U_n \sin \theta_{mn}}
\]
\[
= \sum_{n=0}^{N-1} \frac{h_n}{\cos \theta_{mn} 1 - S_n U_n \sin \theta_{mn}}
\]
\[ (m = 0, 1, \ldots, M - 1) \]
\[ (7) \]

Table 1 Classification of problems

<table>
<thead>
<tr>
<th>No.</th>
<th>Case Name</th>
<th>Acoustic Velocity ( C )</th>
<th>Current Velocity ( U )</th>
<th>Indination Angle ( \theta )</th>
<th>Position of Transponder ( x_\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SUTP1101</td>
<td>given</td>
<td>given</td>
<td>unknown</td>
<td>Given</td>
</tr>
<tr>
<td>2</td>
<td>SUTP1100</td>
<td>given</td>
<td>given</td>
<td>unknown</td>
<td>unknown</td>
</tr>
<tr>
<td>3</td>
<td>SUTP1000</td>
<td>given</td>
<td>unknown</td>
<td>unknown</td>
<td>unknown</td>
</tr>
<tr>
<td>4</td>
<td>SUTP0100</td>
<td>unknown</td>
<td>given</td>
<td>unknown</td>
<td>unknown</td>
</tr>
<tr>
<td>5</td>
<td>SUTP0000</td>
<td>unknown</td>
<td>unknown</td>
<td>unknown</td>
<td>unknown</td>
</tr>
</tbody>
</table>
Horizontal distance between the surface and bottom transponders:

\[ x_{b} - x_{fn} = \sum_{n=0}^{N-1} h_n \tan \theta_{mn}, \quad (m = 0, 1, \ldots, M - 1) \] (8)

The Eqs. (6), (7) and (8) form a closed system. Namely, the number of unknowns \( \theta_{mn} \). \( C_0 \), \( U_0 \), \( x_{n} \), \( m = 0, \ldots, M - 1 \), \( n = 0, \ldots, N - 1 \) is \( MN + 2N + 1 \). On the other hand, the number of equations is \( M(N - 1) + 2M \). Hence, if \( M = 2N + 1 \), (9)

the number of the unknowns coincides with that of the equation.

Snell’s Law given by Eq. (2) can be written as

\[ \frac{C(z) \sin \theta(z) - U(z) \cos \theta_0(z)}{C(z) + U(z) \sin \theta(z)} = k_1(x_F) \] (10)

if the \( x \)-coordinate of the acoustic ray and the constant of the ray are denoted as \( x_F \) and \( k_1(x_F) \) respectively. This equation gives relationship between \( \theta_{mn} \), \( C(z) \), \( U(z) \), and \( k_1(x_F) \). Hence, the unknowns \( \theta_{mn} \) can be replaced by the new unknowns \( k_1(x_F) \). An algorithm to eliminate \( \theta_{mn} \) is given in Appendix.

The nonlinear equations (6) through (8) are solved by Newton-Raphson Method as follows.

Snell’s Law:

\[
\begin{align*}
S(z) & \left( \sin \theta - S_U \sin \theta \cos \theta \right) \\
& \left( 1 + S_U \sin \theta \right) \\
-2S \left( \sin \theta - S_U \sin \theta \cos \theta \right) \\
& \left( 1 + S_U \sin \theta \right) \\
+ \left[ S \left( \cos \theta + 2S_S \sin \theta \right) \\
& \left( 1 + S_U \sin \theta \right) \\
-2S \left( \sin \theta - S_U \sin \theta \cos \theta \right) \\
& \left( 1 + S_U \sin \theta \right) \\
+ \left[ -2S_S \sin \theta \\
& \left( 1 + S_U \sin \theta \right) \\
-2S \left( \sin \theta - S_U \sin \theta \cos \theta \right) \\
& \left( 1 + S_U \sin \theta \right) \\
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{N-1} & h_n \tan \theta_{mn} \sin \theta_{mn} \\
& \sum_{n=0}^{N-1} h_n \tan \theta_{mn} \sin \theta_{mn} \\
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{N-1} & h_n \tan \theta_{mn} \sin \theta_{mn} \\
& \sum_{n=0}^{N-1} h_n \tan \theta_{mn} \sin \theta_{mn} \\
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{N-1} & h_n \tan \theta_{mn} \sin \theta_{mn} \\
& \sum_{n=0}^{N-1} h_n \tan \theta_{mn} \sin \theta_{mn} \\
\end{align*}
\]

Travel time of acoustic signal:

\[
\Delta \tau = \sum_{n=0}^{N-1} \frac{h_n}{S_U \sin \theta_{mn}} + \sum_{n=0}^{N-1} \frac{1}{S_U \sin \theta_{mn}} \\
+ \sum_{n=0}^{N-1} h_n \sin \theta_{mn} \\
+ \sum_{n=0}^{N-1} h_n \cos \theta_{mn} \\
+ \sum_{n=0}^{N-1} h_n \cos \theta_{mn} \\
+ \sum_{n=0}^{N-1} \frac{S_U \sin \theta_{mn}}{S_U \sin \theta_{mn}} \\
+ \sum_{n=0}^{N-1} \frac{S_U \sin \theta_{mn}}{S_U \sin \theta_{mn}} \\
+ \sum_{n=0}^{N-1} \frac{S_U \sin \theta_{mn}}{S_U \sin \theta_{mn}} \\
+ \sum_{n=0}^{N-1} \frac{S_U \sin \theta_{mn}}{S_U \sin \theta_{mn}} \\
\]

Horizontal distance between the surface and bottom transponders:

\[
\begin{align*}
\sum_{n=0}^{N-1} h_n \tan \theta_{mn} & \sum_{n=0}^{N-1} h_n \cos \theta_{mn} \\
\end{align*}
\]

4. Numerical Results in Two-Dimensional Problems

Distribution of underwater sound and current velocity used in numerical calculations are shown in Fig. 3. The water depth is 1000m, and the region is divided into 40 layers.

In the following, Eqs. (11) through (13) are used. First, Problem 2 in Table 1 was solved. In this problem, the sound and current velocities are assumed given. The number of unknowns
\[ \Theta_{mn}, \ x_B: \ m = 1, \ldots, M, \ n = 1, \ldots, N \text{ is } MN + 1, \] and that of equations is \( M(N - 1) + 2M \). Hence, if
\[ M = 1, \]the number of the unknowns coincides that of the equations. When the number of the acoustic rays are bigger than one, the least square procedure is applied.

\[ \text{Fig. 3a} \ \text{Distribution of acoustic velocity } C \]

\[ \text{Fig. 3b} \ \text{Distribution of current velocity } U \]

In the following, five rays are used. The characteristics of the five rays are given in Table 2. The \( x \)-coordinate of the bottom transponder \( x_B \) is assumed zero.

**Table 2**: Characteristics of sound rays

<table>
<thead>
<tr>
<th></th>
<th>Ray 0</th>
<th>Ray 1</th>
<th>Ray 2</th>
<th>Ray 3</th>
<th>Ray 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pos. of Surface</td>
<td>-448.880m</td>
<td>-669.563</td>
<td>-943.529</td>
<td>-1311.899</td>
<td>-1854.256</td>
</tr>
<tr>
<td>Transponder</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One Way Travel</td>
<td>0.744288s</td>
<td>0.817281</td>
<td>0.933793</td>
<td>1.120475</td>
<td>1.431079</td>
</tr>
<tr>
<td>Time</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Incidence Angle</td>
<td>25deg</td>
<td>35</td>
<td>45</td>
<td>55</td>
<td>65</td>
</tr>
<tr>
<td>Ray*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>24.174</td>
<td>33.804</td>
<td>43.334</td>
<td>52.679</td>
<td>61.650</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* The angle between the ray and the negative \( z \) direction at the surface.

\[ \text{Fig. 4a} \ \text{Selection of deceleration coefficients } \alpha \]

\[ \text{Fig. 4b} \ \text{Convergence of ray inclination angles } \Theta \]

\[ \text{Fig. 4c} \ \text{Convergence of horizontal position of bottom transponder } x_B \]

Fig. 4 shows the results for a case where the underwater field is divided into 40 layers. \( \alpha \)s are deceleration coefficients in iteration procedure. As shown in Fig. 4a, \( \alpha \) for \( \Theta_{\text{mn}} \) (i.e. \( \alpha_{\text{T}} \)) and that for \( x_B \) (i.e. \( \alpha_{\text{D}} \)) should be different in general. The convergence of the iteration procedure was very nice in spite of the big number of the layer division. As can be seen from Fig. 4e, the effect of the current velocity on the ray trajectory seems not significant, but the effect can’t be neglected for the precise positioning as premised in the present paper as shown in Fig. 4f.
Fig. 4d Convergence of ray inclination angles $\theta$
(SUTP1100; 40 layers)

Fig. 4e Effect of current velocity on ray trajectory
(SUTP1100; 40 layers; Ray 40 and Ray-1 denote with and without current)

Fig. 4f Difference between ray trajectories with and without current
(SUTP1100; 40 layers; Ray 4)

Fig. 5a Selection of deceleration coefficients $\alpha$
(SUTP1100; 1 layer)

Fig. 5b Convergence of ray inclination angles $\theta$
(SUTP1100; 1 layer; $t[m][n]$ denotes $\theta_{mn}$)

Fig. 5c Convergence of horizontal position of bottom transponder $x_B$
(SUTP1100; 1 layer)

Fig. 5 shows the results for a case where the underwater field is divided into 1 layer. A comparison of the acoustic rays and the horizontal position of the bottom transponder between 40 layers and 1 layer divisions is made in Table 3.

The one layer solution gives a very nice approximation unexpectedly. The average slopes of rays in case of 40 layers division agree very well with those in case of 1 layer division. The horizontal position of the bottom transponder $x_B$ in case of 1 layer division is 0.007m, where the correct value is 0m.
Table 3: A comparison of the acoustic rays and the horizontal position of the bottom transponder between 40 layers and 1 layer divisions (SUTP1100)

<table>
<thead>
<tr>
<th>Incidence</th>
<th>Ray 0</th>
<th>Ray 1</th>
<th>Ray 2</th>
<th>Ray 3</th>
<th>Ray 4</th>
<th>$x_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 layers</td>
<td>25deg</td>
<td>35</td>
<td>45</td>
<td>55</td>
<td>65</td>
<td>0m</td>
</tr>
<tr>
<td>1 layer</td>
<td>24.174</td>
<td>33.804</td>
<td>43.334</td>
<td>52.679</td>
<td>61.650</td>
<td>0m</td>
</tr>
</tbody>
</table>

Table 4: The sound velocity, the current velocity, the rays and the horizontal position of the bottom transponder (SUTP0000; 1 layer)

<table>
<thead>
<tr>
<th>$C$ (m/s)</th>
<th>$U$ (m/s)</th>
<th>Ray 0</th>
<th>Ray 1</th>
<th>Ray 2</th>
<th>Ray 3</th>
<th>Ray 4</th>
<th>$x_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 layer</td>
<td>1473.33</td>
<td>1.4</td>
<td>24.174</td>
<td>33.804</td>
<td>43.334</td>
<td>52.679</td>
<td>61.650</td>
</tr>
<tr>
<td>1 layer</td>
<td>1473.3</td>
<td>1.3</td>
<td>24.183</td>
<td>33.812</td>
<td>43.341</td>
<td>52.687</td>
<td>61.664</td>
</tr>
</tbody>
</table>

Since the sound and current velocities may be unknown in many cases, the results in these situations are shown in Fig. 6. In Fig. 6a, alp_S, alp_T, alp_U and alp_D denote the deceleration coefficients for $s_n$, $u_n$, $\theta_n$ and $x_p$. The sound velocity, the current velocity, the rays and the horizontal position of the bottom transponder are summarized in Table 4. In this calculation, the layer division is one. The results seem to be reasonable, although the precision of $x_p$ is not sufficient. To increase the accuracy, the number of the layers must be increased. This problem will be discussed in the next report.
Fig. 6e Convergence of horizontal position of bottom transponder $x_B$
(SUTP0000; 1 layer)

5. Conclusion

In the previous reports, the current velocity was assumed zero. In the present report, the theory to include the effects of the current was constructed, and some two-dimensional numerical calculations were conducted. According to the results, the effects of the current can't be neglected to realize a high precision positioning of centimeter order necessary to the measurements of the sea bottom crust movements.

To introduce the effects of the currents fully, many studies must be done. When the acoustic and current velocities are unknowns, the most important problem is how to solve the difficulty in convergence of iterations as the number of the layers increases. The results will be reported in the coming reports.

On the other hand, an application of the present theory to a field other than the measurements of the sea bottom crust movements may also be pursued.

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References


Appendix An algorithm to eliminate $\theta_i(z)$

Let $C(z)$, $U(z)$ and $k_i(x_f)$ be given. Then, $\theta_i(z)$ is obtained by solving the two-dimensional form of Eq. (2a):

$$C(z)\sin\theta_i(z) - U_i(z)\cos\theta_i(z) = k_i(x_f)$$

(A1)

Since Eq. (A1) can't be solved analytically, a numerical solution is obtained by using Newton-Raphson method:

$$C(z)\sin\theta_i(z) - U_i(z)\cos\theta_i(z) = k_i(x_f)$$

(A2)

The dependence between $\theta_i(z)$, $C(z)$, $U(z)$ and $k_i(x_f)$ are obtained by differentiating Eq. (A1) as

$$\left\{ \begin{array}{l}
C(z)\cos\theta_i(z) + 2U_i(z)\sin\theta_i(z) \\
-2\left[ C(z)\sin\theta_i(z) - U_i(z)\cos\theta_i(z) \right] \\
U_i(z)\cos\theta_i(z) \\
\sin\theta_i(z)
\end{array} \right\} \cdot \frac{d\theta_i(z)}{C(z) + U_i(z)\sin\theta_i(z)}$$

$$\left\{ \begin{array}{l}
\frac{C(z)\cos\theta_i(z)}{C(z) + U_i(z)\sin\theta_i(z)} \\
\frac{-2\left[ C(z)\sin\theta_i(z) - U_i(z)\cos\theta_i(z) \right]}{C(z) + U_i(z)\sin\theta_i(z)} \\
\frac{\sin\theta_i(z)}{C(z) + U_i(z)\sin\theta_i(z)} \\
\frac{\cos\theta_i(z)}{C(z) + U_i(z)\sin\theta_i(z)}
\end{array} \right\} \cdot \frac{dC(z)}{C(z) + U_i(z)\sin\theta_i(z)}$$

(A3)

Eqs. (A1) and (A3) can be used to eliminate $\theta_i(z)$. 