§8.3　水面に浮かぶ板の運動に関する Hamilton-Kelvin の原理[8.1,2]

図 8.3.1 に座標系 $O(x,y,z)$ その他を示す。板の境界 $\Gamma=\sum_{i=1}^{N}\Gamma_i$ はすべて力学的境界よりなるものとし，境界には外力が働いていないものとする。また，水の密度を ρ とする。

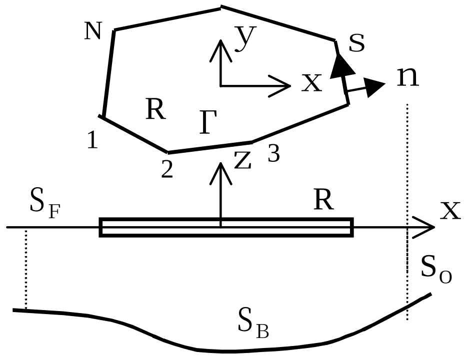

図 8.3.1　水面に浮かぶ板

この場合の運動方程式は，以下の（8.3.1)式と（8.3.2)式で与えられる。

板の力学的条件[M_{plate}]

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+m\omega^2 w-i\omega\rho\phi-\rho g w=0 \quad in\ R \qquad (8.3.1a)$$

$$M_n=0 \quad on\ \Gamma_i, \quad (i=1,2,\cdots,N) \qquad (8.3.1b)$$

$$Q_n+\frac{\partial M_{ns}}{\partial s}=0 \quad on\ \Gamma_i, \quad (i=1,2,\cdots,N) \qquad (8.3.1c)$$

$$M_{ns}=0 \quad at\ P_{i\pm}, \quad (i=1,2,\cdots,N) \qquad (8.3.1d)$$

$$Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \quad in\ R \qquad (8.3.1e)$$

$$Q_y=\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y} \quad in\ R \qquad (8.3.1f)$$

水の力学的条件[M_{water}]

$$u_f=\frac{\partial\phi}{\partial x},\quad v_f=\frac{\partial\phi}{\partial y},\quad w_f=\frac{\partial\phi}{\partial z} \quad in\ \Omega \qquad (8.3.1g)$$

$$\frac{g}{i\omega}\eta = -\phi \quad on\ S_F \tag{8.3.1h}$$

板の運動学的条件$[K_{plate}]$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \quad in\ R \tag{8.3.2a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \quad in\ R \tag{8.3.2b}$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \quad in\ R \tag{8.3.2c}$$

水の運動学的条件$[K_{water}]$

$$\frac{\partial u_f}{\partial x} + \frac{\partial v_f}{\partial y} + \frac{\partial w_f}{\partial z} = 0 \quad in\ \Omega \tag{8.3.2d}$$

$$w_f = i\omega w \quad on\ R \tag{8.3.2e}$$

$$w_f = i\omega\eta \quad on\ S_F \tag{8.3.2f}$$

$$u_f n_x + v_f n_y + w_f n_z = 0 \quad on\ S_B \tag{8.3.2g}$$

$$u_f n_x + v_f n_y + w_f n_z = \bar{f}_O \quad on\ S_O \tag{8.3.2h}$$

（8.3.1a)式と（8.3.2e)式が，板と水との相互影響を与える式である。すなわち，板には流体から圧力$-i\omega\rho\phi - \rho g w$が作用する。また，板と水の$z$方向速度は等しい。また，力学的条件の内，（8.3.1e)式と（8.3.1f)式は補助的な条件で，単にQ_x，Q_yの定義を与えているに過ぎない。

運動学的条件$[K] = [K_{plate}] + [K_{water}]$の下に，仮想変分$\delta w, \delta u_f, \delta v_f, \delta w_f, \delta\eta$を考えると

$$
\begin{aligned}
0 = &-\iint_R \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + m\omega^2 w - i\omega\rho\phi - \rho g w\right)\delta w\, dxdy \\
&+ \sum_{i=1}^{N}\int_{\Gamma_i}\left[-M_n \frac{\partial \delta w}{\partial n} + \left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)\delta w\right] ds \\
&- \sum_{i=1}^{N}[M_{ns}\delta w]_{P_{i+}} + \sum_{i=1}^{N}[M_{ns}\delta w]_{P_{i-}} \\
&+ \rho\iiint_\Omega \left[\left(u_f - \frac{\partial\phi}{\partial x}\right)\delta u_f + \left(v_f - \frac{\partial\phi}{\partial y}\right)\delta v_f + \left(w_f - \frac{\partial\phi}{\partial z}\right)\delta w_f\right] dxdydz \\
&+ \rho\iint_{S_F}\left(\frac{g}{i\omega}\eta + \phi\right)\delta w_f\, dxdy
\end{aligned}
$$

$$= -\omega^2 \iint_R mw\delta w\, dxdy + \iint_R (i\omega\rho\phi + \rho g w)\delta w\, dxdy$$

$$- \iint_R \left[\begin{array}{l} \frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}\delta w\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x}\delta w\right) + \frac{\partial}{\partial x}\left(\frac{\partial M_{xy}}{\partial y}\delta w\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_y}{\partial y}\delta w\right) \\ - \frac{\partial M_x}{\partial x}\frac{\partial \delta w}{\partial x} - \frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y} - \frac{\partial M_{xy}}{\partial y}\frac{\partial \delta w}{\partial x} - \frac{\partial M_y}{\partial y}\frac{\partial \delta w}{\partial y} \end{array} \right] dxdy$$

$$+ \sum_{i=1}^{N} \int_{\Gamma_i} \left[- M_n \frac{\partial \delta w}{\partial n} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \delta w \right] ds$$

$$- \sum_{i=1}^{N} [M_{ns}\delta w]_{P_{i+}} + \sum_{i=1}^{N} [M_{ns}\delta w]_{P_{i-}}$$

$$+ \rho \iiint_\Omega (u_f \delta u_f + v_f \delta v_f + w_f \delta w_f) dxdydz$$

$$- \rho \iiint_\Omega \left[\left(\frac{\partial \phi\, \delta u_f}{\partial x} + \frac{\partial \phi\, \delta v_f}{\partial y} + \frac{\partial \phi\, \delta w_f}{\partial z} \right) - \phi \left(\frac{\partial \delta u_f}{\partial x} + \frac{\partial \delta v_f}{\partial y} + \frac{\partial \delta w_f}{\partial z} \right) \right] dxdydz$$

$$+ \rho \iint_{S_F} \left(\frac{g}{i\omega}\eta + \phi \right) \delta w_f\, dxdy$$

$$= -\omega^2 \iint_R mw\delta w\, dxdy + \iint_R \rho g w \delta w\, dxdy$$

$$- \sum_{i=0}^{N} \int_{\Gamma_i} \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) n_x + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \right) n_y \right] \delta w\, ds$$

$$+ \iint_R \left[\begin{array}{l} \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x} \right) + \frac{\partial}{\partial y}\left(M_{xy} \frac{\partial \delta w}{\partial x} \right) + \frac{\partial}{\partial x}\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) + \frac{\partial}{\partial y}\left(M_y \frac{\partial \delta w}{\partial y} \right) \\ - M_x \frac{\partial^2 \delta w}{\partial x^2} - 2M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} - M_y \frac{\partial^2 \delta w}{\partial y^2} \end{array} \right] dxdy$$

$$+ \sum_{i=1}^{N} \int_{\Gamma_i} \left[- M_n \frac{\partial \delta w}{\partial n} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \delta w \right] ds$$

$$- \sum_{i=1}^{N} [M_{ns}\delta w]_{P_{i+}} + \sum_{i=1}^{N} [M_{ns}\delta w]_{P_{i-}}$$

$$+ \delta \left[\frac{\rho}{2} \iiint_\Omega (u_f^{\,2} + v_f^{\,2} + w_f^{\,2}) dxdydz \right] + \rho g \iint_{S_F} \eta\, \delta\eta\, dxdy$$

$$= -\omega^2 \iint_R mw\delta w\, dxdy + \iint_R \rho g w \delta w\, dxdy$$

$$-\sum_{i=1}^{N}\int_{\Gamma_i}(Q_x n_x + Q_y n_y)\delta w\,ds$$

$$+\sum_{i=1}^{N}\int_{\Gamma_i}\left(M_x n_x \frac{\partial \delta w}{\partial x} + M_{xy} n_x \frac{\partial \delta w}{\partial y} + M_{xy} n_y \frac{\partial \delta w}{\partial x} + M_y n_y \frac{\partial \delta w}{\partial y}\right)ds$$

$$+\iint_R \begin{bmatrix} D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 \delta w}{\partial x^2} + 2(1-\nu)D\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y} \\ + D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 \delta w}{\partial y^2} \end{bmatrix} dxdy$$

$$+\sum_{i=1}^{N}\int_{\Gamma_i}\left[-M_n \frac{\partial \delta w}{\partial n} + \left(Q_n + \frac{\partial M_{ns}}{\partial s}\right)\delta w\right]ds$$

$$-\sum_{i=1}^{N}[M_{ns}\delta w]_{P_{i+}} + \sum_{i=1}^{N}[M_{ns}\delta w]_{P_{i-}}$$

$$+\delta\left[\frac{\rho}{2}\iiint_\Omega (u_f{}^2 + v_f{}^2 + w_f{}^2)dxdydz + \frac{\rho g}{2}\iint_{S_F}\eta^2\,dxdy\right]$$

$$=\delta\left\{\begin{array}{l} -\dfrac{\omega^2}{2}\iint_R mw^2 dxdy \\ +\dfrac{D}{2}\iint_R \begin{bmatrix} \left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu\left(\dfrac{\partial^2 w}{\partial x^2}\right)\left(\dfrac{\partial^2 w}{\partial y^2}\right) + \left(\dfrac{\partial^2 w}{\partial y^2}\right)^2 \\ + 2(1-\nu)\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 \end{bmatrix} dxdy \\ +\dfrac{\rho g}{2}\iint_R w^2 dxdy \end{array}\right\}$$

$$+\delta\left[\frac{\rho}{2}\iiint_\Omega (u_f{}^2 + v_f{}^2 + w_f{}^2)dxdydz + \frac{\rho g}{2}\iint_{S_F}\eta^2\,dxdy\right] \qquad (8.3.3)$$

したがって，(8.3.1)式と（8.3.2)式で与えられる境界値問題は，変分問題

$$\Pi_{HK}[w, \mathbf{u}_f, \eta] = -\frac{\omega^2}{2}\iint_R mw^2 dxdy$$

$$+\frac{D}{2}\iint_R\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]dxdy$$

$$+\frac{\rho g}{2}\iint_R w^2 dxdy$$

$$
\begin{aligned}
&+\frac{\rho}{2}\iiint_{\Omega}(u_f{}^2+v_f{}^2+w_f{}^2)dxdydz \\
&+\frac{\rho g}{2}\iint_{S_F}\eta^2\,dxdy \\
&= stationary \\
&under\quad [K]=[K_{plate}]+[K_{water}]
\end{aligned}
\tag{8.3.4}
$$

と等しい。(8.3.3)式より，この変分問題の自然条件は，力学的条件 $[M]=[M_{plate}]+[M_{water}]$ である。この変分問題のことを，Hamilton-Kelvin の原理と呼ぶことにする。